The filtered Poincaré lemma in higher level (with applications to algebraic groups)

Bernard Le Stum and Adolfo Quirós *

Abstract

We show that the Poincaré lemma we proved elsewhere in the context of crystalline cohomology of higher level behaves well with regard to the Hodge filtration. This allows us to prove the Poincaré lemma for transversal crystals of level m. We interpret the de Rham complex in terms of what we call the Berthelot-Lieberman construction, and show how the same construction can be used to study the conormal complex and invariant differential forms of higher level for a group scheme. Bringing together both instances of the construction, we show that crystalline extensions of transversal crystals by algebraic groups can be computed by reduction to the filtered de Rham complexes. Our theory does not ignore torsion and, unlike in the classical ($m = 0$), not all closed forms are invariant. Therefore, close invariant differential forms of level m provide new invariants and we exhibit some examples as applications.

Contents

Introduction

In a series of articles, starting with [8] and [9], we are using the partial divided powers of Berthelot to study the geometry of algebraic varieties of positive characteristic. This gives new insight into the p-adic cohomological theories. Unlike other works on the subject ([6],

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[12] and [4]), we do not use crystalline cohomology of higher level as a tool to obtain results in rigid cohomology and, in particular, we do not ignore torsion. In fact, torsion is very rich in this theory and provides new invariants that help understand the geometry of algebraic varieties. For example, we will show how the sheaf of closed invariant differential forms of higher level can tell you exactly where the supersingular locus of a family of elliptic curves is.

Following A. Ogus in [10], we introduced in [8] the notion of transversal crystal of higher level. Although we could slightly improve on some of Ogus' results, progress was hampered by the lack of de Rham techniques for computing exactly crystalline cohomology in higher level. A natural answer to this problem was provided in [9], where we developed ideas of P. Berthelot, introducing the de Rham complex of higher level and proving the exact Poincaré lemma. In the present article, we extend the Poincaré lemma of higher level to transversal crystals by paying close attention to filtrations. This is used to give a precise description of the group of extensions of a transversal crystal by a commutative group scheme. In a forthcoming article, we want to use these results to show that Dieudonné crystals of higher level are transversal.

After reviewing in section 1 a few results on filtrations, and especially fixing terminology about filtered derived categories, in section 2 we verify that the formal Poincaré lemma in higher level behaves as expected with respect to filtrations. In doing so, we reinterpret the de Rham and linearized de Rham complexes introduced in [9] as particular cases of what we call a Berthelot-Lieberman complex. This general construction will be used in section 4 to define the conormal complex of higher level of a group scheme. But, before, we prove the filtered Poincaré lemma for transversal crystals. This is done in section 3 and requires careful attention to the behavior of the filtrations all along the process. In section 4 which is completely independent of the preceding one, we study the relation between the conormal sheaf of higher level and invariant differential forms of higher level. Unlike in the classical case (level 0), not all invariant forms are closed. Actually, the module of closed invariant forms is isomorphic to the first cohomology group of the conormal complex. We present concrete examples, including the Legendre family of elliptic curves, and give the relation with de Rham cohomology of higher level in the case of abelian schemes. Section 5 brings together the two previous sections. More precisely, we show that crystalline extensions of transversal crystals by algebraic groups can be computed by reduction to filtered de Rham complexes. As an application, we show that the extension group of the partial divided power ideal by a smooth group is nothing else but a lifting of the module of closed invariant differentials of higher level.

Most results here are inspired by theorems that have been well known for a long time in the case of usual divided powers and classical crystalline cohomology. In particular, this work owes much to P. Berthelot, L. Breen, L. Illusie, W. Messing and A. Ogus.

Conventions

Starting at section 2, we let p be a prime, $m \in \mathbf{N}$ and, unless $m = 0$, all schemes are assumed to be $\mathbf{Z}_{(p)}$-schemes.

1 Generalities about filtrations

Concerning filtrations, we use the terminology of [5], 1.1. In particular, we will only consider filtrations of type $\mathbf{Z}$ in abelian categories. Also, filtration will always mean decreasing filtration.

1.1 Definition

A filtration $\mathrm{Fil}^\bullet$ on an object M is *effective* if $\mathrm{Fil}^0 M = M$.

Unless otherwise specified, we will only consider effective decreasing filtrations. Note that the filtration induced on a subquotient (cf. [5], 1.1.10) by an effective filtration is still effective. One can also check that the image of an effective filtration by a semi-exact (meaning left or right exact) multiadditive functor (cf. [5], 1.1.12) is still effective. This applies in particular to the tensor product filtration.

1.2 Definition

The *trivial (effective) filtration* on an object M is given by

$$\mathrm{Fil}^k M = \begin{cases} M \text{ if } k \leq 0 \\ 0 \text{ if } k > 0 \end{cases}.$$

In [5] (definition 1.3.6), filtered quasi-isomorphisms are only defined for so-called biregular filtrations. This definition does not generalize well. As in [10], section 4.4, we will need a more restrictive notion.

1.3 Definition

A morphism of filtered complexes $M^\bullet \to N^\bullet$ is a *true filtered quasi-isomorphism* if, for each $k \in \mathbf{Z}$, the induced morphism $\mathrm{Fil}^k M^\bullet \to \mathrm{Fil}^k N^\bullet$ is a quasi-isomorphism. A *filtered homotopy* is a morphism of filtered complexes which is a homotopy. Two morphisms of filtered complexes $f, g : M^\bullet \to N^\bullet$ are *homotopic* if there exists a filtered homotopy $h : M^\bullet \to N^\bullet$ such that g is homotopic to f with respect to h.

As explained in [10], page 80, the above notions of filtered homotopy and true filtered quasi-isomorphisms are suitable to define the filtered derived category of a Grothendieck category, e.g. a category of modules. Moreover, any left exact additive functor F between

such categories gives rise to a filtered derived functor and this construction is completely compatible with the non-filtered situation in the sense that we always have

$$RF(\mathrm{Fil}^k M^\bullet) = \mathrm{Fil}^k RFM^\bullet.$$

In particular, there exists a canonical spectral sequence

$$E_1^{i,j} = R^{i+j}F(Gr^i M^\bullet) \Rightarrow R^n F(M^\bullet)$$

that endows, for each $n \in \mathbf{Z}$, $R^n F(M^\bullet)$ with a canonical filtration.

We now need to recall some definitions and results from [10] and [8] on transversal filtrations.

1.4 Definition

A *ring filtration* on a ring A is an effective filtration by ideals $I^{(k)}$ such that $I^{(k)}I^{(l)} \subset I^{(k+l)}$. A filtered A-module $(M, \mathrm{Fil}^\bullet)$ is *transversal* (resp. *almost transversal*) to $I^{(\bullet)}$ if for each $k \in \mathbf{Z}$,

$$I^{(1)}M \cap \mathrm{Fil}^k M = (resp. \subset) \sum_{i+j=k, i>0} I^{(i)} \mathrm{Fil}^j .$$

If $(M, \mathrm{Fil}^\bullet)$ is a filtered A-module, the *saturation* of the filtration with respect to $I^{(\bullet)}$ is the tensor product filtration $\overline{\mathrm{Fil}}^\bullet$ on M under the identification $A \otimes_A M = M$.

When A is filtered by the powers of an ideal I, we simply say transversal (resp. almost transversal, resp. saturation with respect) to I. Note that a module filtration is always transversal to 0 and that, if a filtration is almost transversal, then its saturation is transversal.

1.5 Definition

The *trivial transversal filtration* on an A-module M is the saturation of the trivial filtration.

Note that the trivial transversal filtration is given by

$$\overline{\mathrm{Fil}}^k M = \begin{cases} M \text{ if } k \leq 0 \\ I^{(k)}M \text{ if } k > 0 \end{cases}$$

2 The formal filtered Poincaré lemma

The aim of this section is to give a filtered version of theorem 3.3 of [9]. We first recall some properties of m-PD-envelopes and introduce the notion of Berthelot-Lieberman complex.

If $X \hookrightarrow Y$ is an immersion of schemes, then we will denote by $P_{Xm}(Y)$ its divided power envelope of level m. We will write $\mathcal{P}_{Xm}(Y)$ for the structural sheaf and $\mathcal{I}_{Xm}(Y)$ the m-PD-ideal. We will use a superscript n to denote the same objects modulo $\mathcal{I}_{Xm}^{\{n+1\}}(Y)$. If X is an S-scheme, we will denote by $P_{X/Sm}(r)$ the m-PD envelope of the diagonal embedding of X in X^{r+1} and modify all other notations accordingly.

The notion of m-PD envelope is functorial in the sense that any commutative diagram

$$\begin{array}{ccc} X' & \hookrightarrow & Y' \\ \downarrow g & & \downarrow g' \\ X & \hookrightarrow & Y \end{array}$$

canonicaly extends to

$$\begin{array}{ccccc} X' & \hookrightarrow & P_{X'm}(Y') & \hookrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & P_{Xm}(Y) & \hookrightarrow & Y \end{array}$$

We recall the following fundamental results of Berthelot :

2.1 Proposition

Let $f : X \hookrightarrow Y$ be an immersion of schemes. Then,

1. *If f has locally a smooth retraction then, for all n, $\mathcal{P}^n_{Xm}(Y)$ is a locally free $\mathcal{O}_X$-module of finite rank. Actuallly, $\mathcal{P}_{Xm}(Y)$ itself is locally free when Y is uniformly killed by a power of p.*

2. *Assume that f has a smooth retraction $p : Y \to X$ and let*

$$\begin{array}{ccc} X' & \overset{\leftarrow}{\hookrightarrow} & Y' \\ \downarrow g & & \downarrow g' \\ X & \overset{\leftarrow}{\hookrightarrow} & Y \end{array}$$

be a be cartesian diagram (both ways). Then, we have for all n,

$$g^*\mathcal{P}^n_{Xm}(Y) \simeq \mathcal{P}^n_{X'm}(Y').$$

and even

$$g^*\mathcal{P}_{Xm}(Y) \simeq \mathcal{P}_{X'm}(Y').$$

if Y is uniformly killed by a power of p.

Proof : This follows from propositions 1.4.6 and 1.5.3 of [2].

Let $X \hookrightarrow Y(\bullet)$ be an immersion of a constant scheme into a simplicial complex. In other words, we are given a family of immersions of $X \hookrightarrow Y(r)$ compatible with the differentials d_i and the degeneracy arrows s_i of $Y(\bullet)$. Taking m-PD envelopes gives rise to a simplicial complex $P_{Xm}(Y\bullet)$ from which we derive a complex $(\mathcal{P}_{Xm}(Y\bullet), \mathcal{I}^{\{k\}}_{Xm}(Y\bullet))$ of filtered rings. We then consider as in [7] the normalization of $\mathcal{P}_{Xm}(Y\bullet)$ which is the subcomplex of ideals defined by $N\mathcal{P}_{Xm}(Yr) := \cap \ker s_i^*$. Finally, we are interested in the quotient of $N\mathcal{P}_{Xm}(Y\bullet)$ by the differential subalgebra generated by the ideal $\mathcal{I}^{\{p^m+1\}}_{Xm}(Y1)$ of $\mathcal{P}_{Xm}(Y1)$, which we write $\Omega^\bullet_{Xm}(Y)$.

2.2 Definition

The complex $\Omega^\bullet_{Xm}(Y)$ is the *Berthelot-Lieberman complex* of $X \hookrightarrow Y(\bullet)$. The *Hodge filtration* on $\Omega^\bullet_{Xm}(Y)$ is the filtration $\mathrm{Fil}^\bullet_H$ induced by the filtration of $\mathcal{P}_{Xm}(Y\bullet)$.

2.3 Definition

1. In a category with products, the *product simplicial complex* $X^{prod}(\bullet)$ of an object X is defined by $X^{prod}(r) = X^{r+1}$,

$$d_i(x_1, \dots, x_{r+2}) = (x_1, \dots, x_i, x_{i+2}, \dots, x_{r+2})$$

and

$$s_i(x_1, \dots, x_r) = (x_1, \dots, x_i, x_{i+1}, x_{i+1}, x_{i+2}, \dots, x_{r+2}).$$

2. If $X(\bullet)$ is a simplicial complex in any category, the *shifted simplicial complex* is defined by $X^+(r) = X(r+1), d_i^+ = d_{i+1}, s_i^+ = s_{i+1}$.

We can now reformulate some definitions from section 1 of [9].

Let S be a scheme with p locally nilpotent and X an S-scheme.

2.4 Definition

The Berthelot-Lieberman complex $\Omega^\bullet_{X/Sm}$ of the diagonal embedding $X \hookrightarrow (X/S)^{prod}(\bullet)$, is the *de Rham complex of level* m. The Berthelot-Lieberman complex $L_X(\Omega^\bullet_{X/Sm})$ of the shifted simplicial complex $(X/S)^{prod+}(\bullet)$ is the *linearized de Rham complex of level* m.

Note that when X is smooth, the de Rham complex of level 0 is the usual de Rham complex with its Hodge filtration. Note also that $L_X(\Omega^\bullet_{X/Sm})$ is what we called $\Omega^\bullet_{P/Sm}$ in [9].

For the rest of this section, we assume that X is a smooth scheme over S.

2.5 Remarks

Recall from section 1 of [9] that, if we have local coordinates $t_1, \dots, t_n$ on X and, as usual, we set $\tau_i := 1 \otimes t_i - t_i \otimes 1$, then $\mathcal{P}_{X/Sm}(r)$ is a free $\mathcal{O}_X$-module on generators $\tau^{\{J_1\}} \otimes \cdots \otimes \tau^{\{J_r\}}$ with $|J_i| \geq 0$ (we use the standard multiindex convention). Using proposition 1.5.3 of [2], one easily checks that $\mathcal{I}^{\{k\}}_{X/Sm}(r)$ is generated by the $\tau^{\{J_1\}} \otimes \cdots \otimes \tau^{\{J_1\}}$ with $|J_1| + \cdots |J_r| \geq k$.

Now, if as usual again, dt_i denotes the image of τ_i in $\Omega^1_{X/Sm}$, we know that $\Omega^1_{X/Sm}$ is a free $\mathcal{O}_X$-module on generators $(dt)^J$ with $0 < |J| \leq p^m$. As we discussed in section 1 of [9], even if $\Omega^r_{X/Sm}$ is generated by the $(dt)^{J_1} \otimes \cdots \otimes (dt)^{J_r}$ with $0 < |J_i| \leq p^m$, there are some relations among these generators. Anyway, we see that $\mathrm{Fil}^k_H \Omega^r_{X/Sm}$ is generated by the $(dt)^{J_1} \otimes \cdots \otimes (dt)^{J_1}$ with $0 < |J_i| \leq p^m$ and $|J_1| + \cdots |J_r| \geq k$. In particular, $\mathrm{Fil}^k_H = 0$ for $k > rp^m$.

Finally $L_X(\Omega^r_{X/Sm})$ is generated as $\mathcal{P}_{X/Sm}(1)$-module by the $(d\tau)^{J_1} \otimes \cdots \otimes (d\tau)^{J_r}$ with $0 < |J_i| \leq p^m$, and $\mathrm{Fil}^k_H L_X(\Omega^r_{X/Sm})$ is the $\mathcal{O}_X$-submodule generated by the $\tau^{\{I\}}(d\tau)^{J_1} \otimes \cdots \otimes (d\tau)^{J_r}$ with $|I| + |J_1| + \cdots |J_r| \geq k$.

A quick look at the proof of proposition 1.4 in [9] shows that the Hodge filtration is a filtration by locally free submodules. In fact, this question is local and we may then proceed by extracting a basis from a given set of generators. But all the relations are homogeneous and therefore, respect the Hodge filtration.

Recall that if $\mathcal{F}$ is any $\mathcal{D}^{(m)}_{X/S}$-module, we can form the de Rham complex

$$\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Sm}.$$

of $\mathcal{F}$. In particular, we have the *m-connection* $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/Sm}$. For further use, recall also that the de relative de Rham cohomology of level m of $\mathcal{F}$ is

$$\mathcal{H}^n_{dRm}(X, \mathcal{F}) := \mathbf{R}^n p_{X*} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Sm}$$

where $p_X : X \to S$ si the structural map.

Concerning Griffiths tranversality, we refer to section 2.2 of [8]. Let us just recall that a filtration $\mathrm{Fil}^\bullet$ on a $\mathcal{D}^{(m)}_{X/S}$-module $\mathcal{F}$ is a filtration by $\mathcal{O}_X$-submodules and that it is said *Griffiths tranvsersal* if we always have

$$\mathcal{D}^{(m)}_{X/Sj} \mathrm{Fil}^k \mathcal{F} \subset \mathrm{Fil}^{k-j} \mathcal{F}.$$

2.6 Proposition

ÊA *filtered $\mathcal{D}^{(m)}_{X/S}$-module is Griffiths transversal if and only if the m-connection is compatible with the filtrations.*

Proof : This is a local question and, if we have local coordinates, the m-connection on $\mathcal{F}$ is given by

$$s \mapsto \sum_{0<|J|\leq p^m} \partial^{[J]} s \otimes (dt)^J.$$

If the filtration is Griffiths transversal and $s \in \mathrm{Fil}^k \mathcal{F}$, we have

$$\partial^{[J]} s \otimes (dt)^J \in \mathrm{Fil}^{k-|J|} \mathcal{F} \otimes \mathrm{Fil}^{|J|} \Omega^\bullet_{X/Sm} \subset \mathrm{Fil}^k[\mathcal{F} \otimes \Omega^\bullet_{X/Sm}]$$

It follows that the connection preserves the filtration. Conversely, if the m-connection is compatible with the filtration and $s \in \mathrm{Fil}^k \mathcal{F}$, then, necessarily, for all $i = 1, \ldots, n$ and $j \leq m$, we have

$$\partial^{[p^j]} s \otimes (dt)^{p^j} \in \mathrm{Fil}^k[\mathcal{F} \otimes \Omega^\bullet_{X/Sm}]$$

so that, necessarily, $\partial_i^{[p^j]} s \in \mathrm{Fil}^{k-|p^j|} \mathcal{F}$ and by remark 2.2.2 (ii) of [8], this is sufficient for Griffiths tranversality.

Since the differential on a de Rham complex can be described by the Leibnitz rule, we have the following :

2.7 Corollary

If $\mathcal{F}$ is a filtered $\mathcal{D}^{(m)}_{X/S}$-module and if, for each r, $\mathcal{F} \otimes \Omega^r_{X/Sm}$ is endowed with the tensor product filtration, then $\mathcal{F}$ is Griffiths transversal if and only if the de Rham complex of $\mathcal{F}$ is a filtered complex.

2.8 Lemma

For all r, if we endow $\Omega^r_{X/Sm}$ and $L_X(\Omega^r_{X/Sm})$ with their Hodge filtration and $\mathcal{P}_{X/Sm}(1)$ with its m-PD-filtration, we have an isomorphism of filtered modules

$$L_X(\Omega^r_{X/Sm}) = \mathcal{P}_{X/Sm}(1) \otimes_{\mathcal{O}_X} \Omega^r_{X/Sm}.$$

Moreover, the filtration on $\mathcal{P}_{X/Sm}(1) \otimes_{\mathcal{O}_X} \Omega^r_{X/Sm}$ is the saturation with respect to $\mathcal{I}^{\{\bullet\}}_{X/Sm}(1)$ of the inverse image by $\mathcal{P}_{X/Sm}(1) \to \mathcal{O}_X$ of the Hodge filtration on $\Omega^r_{X/Sm}$.

Proof : Since we assume that X is smooth over S, then as mentioned in section 1.5 of [9], $\mathcal{P}^+_{X/Sm}(r)$ is canonically isomorphic as filtered algebra to $\mathcal{P}_{X/Sm}(1) \otimes_{\mathcal{O}_X} \mathcal{P}_{X/Sm}(r)$. The assertion then follows from the functoriality of our construction. The second assertion is local in nature and follows directly from the above local description of our filtrations.

2.9 Proposition

The Hodge filtration on $L_X(\Omega^\bullet_{X/Sm})$ is transversal to $\mathcal{I}^{\{\bullet\}}_{X/Sm}(1)$.

Proof : The morphism of filtered rings $(\mathcal{P}_{X/Sm}(1), \mathcal{I}^{\{\bullet\}}_{X/Sm}(1)) \to (\mathcal{O}_X, 0)$ obviously satisfies the assumptions of proposition 1.1.8 of [8]. Since any filtration is transversal to the 0-ideal, we see that the inverse image of the Hodge filtration is almost transversal and it follows that its saturation is transversal. Our assertion now results from lemma 2.8

The following is a generalization of theorem 3.3 of [9]

2.10 Proposition

If $\mathcal{O}_X$ is endowed with the trivial filtration and $L_X(\Omega^\bullet_{X/Sm})$ with the Hodge filtration, the canonical map

$$\mathcal{O}_X \to L_X(\Omega^\bullet_{X/Sm})$$

is a true filtered quasi-isomorphism. More precisely, locally on X, it is a filtered homotopy equivalence.

Proof : The proof works exactly as in [9] once one notices that the homotopy of 2.1 in [9] is a filtered homotopy.

3 The filtered Poincaré lemma

We will explain here how the results of [9], section 4 extend to the case of transversal m-crystals.

Let $(S, \mathfrak{a}, \mathfrak{b})$ be a m-PD-scheme with p locally nilpotent and $p \in \mathfrak{a}$. Let X be an S-scheme to which the m-PD-structure of S extends.

Unless otherwise specified, we will assume in this section that X is smooth over S.

3.1 Notations

If Y is any object in a topos $\mathcal{T}$, we will denote by $\mathcal{T}_{/Y}$ the localized category and by $j_Y : \mathcal{T}_{/Y} \to \mathcal{T}$ the restriction map.

In particular, we will consider here the restriction map

$$j_X : (X/S)^{(m)}_{\mathrm{cris}\,|X} \to (X/S)^{(m)}_{\mathrm{cris}}$$

and the projection

$$u_X : (X/S)^{(m)}_{\mathrm{cris}} \to X_{Zar}.$$

3.2 Proposition

The ideal $\mathcal{K}^{(m)}_{X/S} := j_{X*}j_X^{-1}\mathcal{I}^{(m)}_{X/S}$ *of* $j_{X*}j_X^{-1}\mathcal{O}^{(m)}_{X/S}$ *is an* m*-PD-ideal, there is a canonical exact sequence*

$$0 \longrightarrow \mathcal{K}^{(m)}_{X/S} \longrightarrow j_{X*}j_X^{-1}\mathcal{O}^{(m)}_{X/S} \longrightarrow u_X^{-1}\mathcal{O}_X \longrightarrow 0$$

and for all $k \in \mathbf{Z}$*, we have* $\mathcal{K}^{(m)\ \{k\}}_{X/S} = j_{X*}j_X^{-1}[\mathcal{I}^{(m)\ \{k\}}_{X/S}]$.

Proof : One easily checks that, if $\mathcal{F}$ is any $\mathcal{O}_X$-module, the adjunction map

$$j_{X*}j_X^{-1}u_X^{-1}\mathcal{F} \to u_X^{-1}\mathcal{F}$$

is an isomorphism. In particular, if we apply the exact functor $j_{X*}j_X^{-1}$ to the exact sequence

$$0 \longrightarrow \mathcal{I}^{(m)}_{X/S} \longrightarrow \mathcal{O}^{(m)}_{X/S} \longrightarrow u_X^{-1}\mathcal{O}_X \longrightarrow 0,$$

we get the expected exact sequence

$$0 \longrightarrow j_{X*}j_X^{-1}\mathcal{I}^{(m)}_{X/S} \longrightarrow j_{X*}j_X^{-1}\mathcal{O}^{(m)}_{X/S} \longrightarrow u_X^{-1}\mathcal{O}_X \longrightarrow 0.$$

Thus, we see that if $U \subset X$ is any open subset and $U \hookrightarrow Y$ an m-PD-thickening, we have the following exact sequence

$$0 \longrightarrow (K^{(m)}_{X/S})_Y \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_U \longrightarrow 0$$

where $P := P_{Um}(Y \times_S X)$. Hence, we see that $(K^{(m)}_{X/S})_Y$ is the m-PD-ideal $\mathcal{I}_P$ of P. It follows that $K^{(m)}_{X/S}$ is an m-PD-ideal and that for all $k \in \mathbf{Z}$, $(K^{(m)\ \{k\}}_{X/S})_Y = \mathcal{I}_P^{\{k\}} = (\mathcal{I}^{(m)\ \{k\}}_{X/S})_Y$.

Recall that $u_{|X} := u_X \circ j_X$ is in a natural way a morphism of ringed sites.

3.3 Definition

If $\mathcal{F}$ is a filtered $\mathcal{O}_X$-module, its *linearization (of level m)* is $L^{(m)}(\mathcal{F}) := j_{X*}u^*_{|X}\mathcal{F}$ endowed with the saturation of the filtration $L^{(m)}(\mathrm{Fil}^k\mathcal{F})$ with respect to the m-PD-ideal $\mathcal{K}_{X/S}^{(m)}{}^{\{\bullet\}}$ of $L(\mathcal{O}_{X/S}^{(m)})$.

3.4 Lemma

If $\mathcal{F}$ is a filtered $\mathcal{O}_X$-module,we have a canonical isomorphism of filtered modules

$$L^{(m)}(\mathcal{F})_X = \mathcal{P}_{X/Sm}(1) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Proof : Thanks to proposition 4.3 (1) of [9], only the assertion concerning the filtration has to be checked. Since saturation is just tensor product with the ideal filtration, it is sufficient to note that, as in the proof of lemma 3.2, $(\mathcal{K}_{X/S}^{(m)})_X := \mathcal{I}_{X/Sm}(1)$.

We recall that $L^{(m)}(\Omega^\bullet_{X/Sm})$ has a natural structure of complex of crystals whose realization on X is nothing but the linearized de Rham complex $L_X(\Omega^\bullet_{X/Sm})$. We can now state the formal Poincaré lemma in its crystalline form.

3.5 Theorem

If E is a filtered $\mathcal{O}_{X/S}^{(m)}$-module which is saturated with respect to $\mathcal{I}_{X/S}^{(m)}$, then the morphism

$$E \to E \otimes_{\mathcal{O}_{X/S}^{(m)}} L^{(m)}(\Omega^\bullet_{X/Sm})$$

is a true filtered quasi-isomorphism. More precisely, locally on $\mathrm{Cris}^{(m)}(X/S)$*, it is a filtered homotopy equivalence.*

Proof : Since the filtration on E is saturated, the canonical identification

$$E \otimes_{\mathcal{O}_{X/S}^{(m)}} \mathcal{O}_{X/S}^{(m)} = E$$

is compatible with the filtrations. Since the pull-back of a filtered homotopy is still filtered homotopy, the standard arguments allow us to reduce to the case $E = \mathcal{O}_{X/S}^{(m)}$ and then to theorem 2.10

The definition of a transversal m-crystal is given in [8], section 4. Roughly speaking, it is a crystal of transversal modules, but the reader should consider looking at the above reference if he really wants a precise definition as well as a description of its relation with Griffiths tranversality.

3.6 Proposition

Let E be a transversal m-crystal on X/S. If $\mathcal{F}$ is any filtered $\mathcal{O}_X$-module, there is a canonical isomorphism of filtered modules

$$E \otimes L^{(m)}(\mathcal{F}) \simeq L^{(m)}(E_X \otimes \mathcal{F}).$$

Proof : We already have an isomorphism of crystals thanks to proposition 4.3 (3) of [9]. More precisely, if If $U \subset X$ is any open subset and $U \hookrightarrow Y$ an m-PD-thickening, we have an isomorphism

$$E_Y \otimes \mathcal{O}_P \otimes \mathcal{F} \simeq E_P \otimes \mathcal{F} \simeq \mathcal{O}_P \otimes E_Y \otimes \mathcal{F},$$

where $P := P_{Um}(Y \times_S X)$ as in the proof of proposition 3.2, and we need to show that it is compatible with filtrations. But this follows from the definition in [8], 4.2.2 of a transversal m-crystal.

3.7 Corollary

If E is a transversal m-crystal on X/S, there is a canonical true filtered quasi-isomorphism

$$E \to L^{(m)}(E_X \otimes_{\mathcal{O}_X} \Omega^{\bullet}_{X/Sm}).$$

More precisely, locally on $\mathrm{Cris}^{(m)}(X/S)$*, it is a filtered homotopy equivalence.*

In order to obtain the filtered Poincaré lemma at the cohomological level, we need the following :

3.8 Proposition

If $\mathcal{F}$ is a filtered $\mathcal{O}_X$-module, we have a canonical isomorphism in the filtered derived category $Ru_{X*}L^{(m)}(\mathcal{F}) = \mathcal{F}$.

Proof : We must show that, for all $k \in \mathbf{Z}$, we have $R^i u_{X*} \operatorname{Fil}^k L^{(m)}(\mathcal{F}) = 0$ for $i > 0$ and $u_{X*} \operatorname{Fil}^k L^{(m)}(\mathcal{F}) = \operatorname{Fil}^k \mathcal{F}$. Using proposition 3.2,

$$\operatorname{Fil}^k L^{(m)}(\mathcal{F}) = \sum_{i+j=k, i\geq 0} (j_{X*} j_X^{-1} \mathcal{I}_{X/S}^{(m)\,\{i\}})(j_{X*} u_{|X}^{*}(\operatorname{Fil}^j \mathcal{F})).$$

Since j_{X*} is exact, it follows that $\operatorname{Fil}^k L^{(m)}(\mathcal{F}) = j_{X*}E$ with

$$E := \sum_{i+j=k, i\geq 0} j_X^{-1} \mathcal{I}_{X/S}^{(m)\,\{i\}} u_{|X}^{*}(\operatorname{Fil}^j \mathcal{F}).$$

Regarding the higher direct images, it is sufficient to recall, that since j_{X*} and $u_{|X*}$ are exact, it is automatic that $R^i u_{X*} j_{X*} E = 0$ for $i > 0$.
Now, we know from proposition 4.3 (2) of [9] that, for any $\mathcal{O}_X$-module $\mathcal{F}$, we have $u_{X*} L^{(m)}(\mathcal{F}) = \mathcal{F}$. Also, u_X^{-1} being fully faithful, $u_{X*} u_X^{-1} \mathcal{F} = \mathcal{F}$. Since u_{X*} is exact and

$\mathcal{K}^{(m)}_{X/S}$ is the kernel of the natural map $L^{(m)}(\mathcal{O}_X) \to u_X^{-1}\mathcal{O}_X$, we see that $u_{X*}\mathcal{K}^{(m)}_{X/S} = 0$. It follows that u_{X*} ignores saturation with respect to $\mathcal{K}^{(m)}_{X/S}$ and so,

$$u_{X*} \operatorname{Fil}^k L^{(m)}(\mathcal{F}) = u_{X*}L^{(m)}(\operatorname{Fil}^k \mathcal{F}) = \operatorname{Fil}^k \mathcal{F}$$

as asserted.

3.9 Theorem

If E is a transversal m-crystal on X/S, there is a canonical isomorphism in the filtered derived category

$$Ru_{X*}E \simeq E_X \otimes \Omega^\bullet_{X/Sm}.$$

Proof : Using proposition 3.8, this is an direct consequence of 3.7.

3.10 Corollary

Even if we no longer assume X smooth, but if $i : X \hookrightarrow Y$ is an embedding into a smooth S-scheme and if E is a transversal m-crystal on X, then there is a canonical isomorphism in the filtered derived category

$$i_*Ru_{X*}E \simeq (i_{\mathrm{cris}\,*}E)_Y \otimes \Omega^\bullet_{Y/Sm}.$$

Proof : Works exactly as at the end of section 4 in [9].

3.11 Corollary

In the situation of the previous corollary, there is a canonical filtered isomorphism

$$R\Gamma((X/S)^{(m)}_{\mathrm{cris}}, E) \simeq R\Gamma(Y, (i_{\mathrm{cris},m*}E)_Y \otimes \Omega^\bullet_{Y/Sm})).$$

In other words, for all i, we have

$$H^i_{\mathrm{cris},m}(X, \operatorname{Fil}^k E) = H^i(\operatorname{Fil}^k((i_{\mathrm{cris},m*}E)_Y \otimes \Omega^\bullet_{Y/Sm})).$$

4 Differentials of higher level on a group scheme

In this section, which is independent of the previous one, we define the conormal complex of level m of a group scheme and study invariant differential forms of higher level.

4.1 Definition

The *simplicial complex $G^{gr}(\bullet)$ associated to a group G* in a category with products has components $G^{gr}(r) := G^r$, differentials $d_i : G^{r+1} \to G^r$ defined by

$$\begin{array}{rcl} d_0(g_1, \ldots, g_{r+1}) & = & (g_2, \ldots, g_{r+1}) \\ & \vdots & \\ d_i(g_1, \ldots, g_{r+1}) & = & (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{r+1}) \\ & \vdots & \\ d_{r+1}(g_1, \ldots, g_{r+1}) & = & (g_1, \ldots, g_r) \end{array}$$

and degeneracy arrows $s_i : G^{r-1} \to G^r$ defined by

$$s_i(g_1, \ldots, g_{r-1}) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_{r-1}).$$

4.2 Lemma

If G is a group in a category with products, the maps

$$G^{r+1} \to G^r : (g_1, \ldots, g_{r+1}) \mapsto (g_1^{-1} g_2, g_2^{-1} g_3, \ldots, g_r^{-1} g_{r+1})$$

define a morphism of simplicial complexes $\nu(\bullet) : G^{prod}(\bullet) \to G^{gr}(\bullet)$.

Proof : This is easily checked.

Let G be a group scheme over a scheme S, $p_G : G \to S$ its structural morphism, $1_G : S \hookrightarrow G$ its unit section and

$$\mu, p_1, p_2 : G \times_S G \to G.$$

the group law and the projections.

4.3 Definition

The *conormal complex of level m* of G is the Berthelot-Lieberman filtered complex $\omega^{\bullet}_{Gm}$ associated to the unit embedding of S into $G^{gr}(\bullet)$.

It will be convenient to write $P_{1m}(r)$ for the m-PD-envelope of the unit section in G^r and modify the other notations accordingly.

4.4 Remark

As in the case of the de Rham complex, for which the results are recalled in remark 2.5, we have a very simple local description of the situation in the case of a smooth group scheme. If $s_1, \ldots, s_n$ is a regular sequence of local parameters for the unit section, then $\mathcal{P}_{1m}(r)$ is a free module on generators $s^{\{J_1\}} \otimes \cdots \otimes s^{\{J_r\}}$ with $|J_i| \geq 0$ and $\mathcal{I}^{\{k\}}_{1m}(r)$ is generated by the $s^{\{J_1\}} \otimes \cdots \otimes s^{\{J_1\}}$ with $|J_1| + \cdots |J_r| \geq k$. Thus, if $\bar{s}_i$ denotes the image of s_i in ω^1_{Gm}, we see that ω^1_{Gm} is a free module on generators $\bar{s}^J$ with $0 < |J| \leq p^m$ and that, for bigger r, ω^r_{Gm} is generated by the $\bar{s}^{J_1} \otimes \cdots \otimes \bar{s}^{J_r}$ with $0 < |J_i| \leq p^m$, subject to some relations. Of course, the k-th step of the Hodge filtration $\mathrm{Fil}^k\, \omega^r_{Gm}$ has the same generators subject to the additional condition that $|J_1| + \cdots |J_r| \geq k$. In particular, $\mathrm{Fil}^r\, \omega^r_{Gm} = \omega^r_{Gm}$.

4.5 Proposition

There exist canonical morphisms of filtered complexes

$$\nu(\bullet) : \mathcal{P}_{1m}(\bullet) \to p_{G*}\mathcal{P}_{Gm}(\bullet)$$

and

$$\nu^{\bullet} : \omega^{\bullet}_{Gm} \to p_{G*}\Omega^{\bullet}_{Gm}.$$

Proof : Using lemma 4.2, this follows from the functoriality of the construction of the Berthelot-Lieberman complex.

4.6 Remark

Of course, we also have for each r morphisms of filtered modules

$$\nu(r) : p_G^*\mathcal{P}_{1m}(r) \to \mathcal{P}_{Gm}(r)$$

and

$$\nu^r : p_G^*\omega^r_{Gm} \to \Omega^r_{Gm}.$$

There are also morphisms of filtered complexes

$$\mathcal{P}_{1m}(\bullet) = 1_G^* p_G^* \mathcal{P}_{1m}(\bullet) \to 1_G^* p_G^* p_{G*}\mathcal{P}_{Gm}(\bullet) \to 1_G^*\mathcal{P}_{Gm}(\bullet)$$

and in particular

$$\omega^{\bullet}_{Gm} \to 1_G^*\Omega^{\bullet}_{Gm}.$$

Without assuming that G is smooth, we cannot really say more.

If $g \in G(S)$, we denote by $T_g : G \to G$ the left translation map given by $h \mapsto gh$.

4.7 Definition

A section ξ of $\mathcal{P}_{Gm}(r)$, or $\mathcal{P}^k_{Gm}(r)$ is *invariant by translation by* $g \in G(S)$ if $T_g^*(\xi) = \xi$. Moreover, ξ is *invariant by translation* if it is invariant by translation by any section of $G_{S'}$ for any extension $S' \to S$ of the basis.

We indicate with a superscript $\bullet^{inv}$ the subsheaf of translation invariant sections.

4.8 Proposition

If G is smooth, we have for all r a canonical isomorphism of filtered modules

$$p_G^*\mathcal{P}_{1m}(r) \simeq \mathcal{P}_{Gm}(r)$$

and therefore also

$$p_G^*\omega^r_{Gm} \simeq \Omega^r_{Gm}.$$

We also have canonical isomorphisms of filtered complexes

$$\mathcal{P}_{1m}(\bullet) \simeq 1_G^*\mathcal{P}_{Gm}(\bullet) \simeq \mathcal{P}_{Gm}^{inv}(\bullet)$$

and therefore also

$$\omega_{Gm}^\bullet \simeq 1_G^*\Omega_{Gm}^\bullet \simeq \Omega_{Gm}^{\bullet inv}.$$

Proof : Since G is smooth, it follows from the second assertion of proposition 2.1 that, for all r, we have a canonical isomorphism of filtered modules

$$p_G^*\mathcal{P}_{1m}(r) \simeq \mathcal{P}_{Gm}(r).$$

The functoriality of the Berthelot-Lieberman construction provides us with a canonical isomorphism of filtered complexes

$$\mathcal{P}_{1m}(\bullet) \simeq 1_G^*\mathcal{P}_{Gm}(\bullet).$$

We also have injective maps

$$\mathcal{P}_{1m}(r) \hookrightarrow p_{G*}\mathcal{P}_{Gm}(r).$$

Moreover, since for $g \in G(S)$ we have

$$\nu(r) \circ (T_g \times \cdots \times T_g) = \nu(r) : G^{r+1} \longrightarrow G^r,$$

we see that the image of $\mathcal{P}_{1m}(\bullet)$ in $p_{G*}\mathcal{P}_{Gm}(\bullet)$ is contained in $\mathcal{P}_{Gm}^{inv}(\bullet)$. We want to show that the map

$$\mathcal{P}_{1m}(\bullet) \hookrightarrow \mathcal{P}_{Gm}^{inv}(\bullet)$$

is bijective. Since

$$p_{G*}\mathcal{P}_{Gm}(r) \simeq p_{G*}\mathcal{O}_G \otimes_{\mathcal{O}_S} \mathcal{P}_{1m}(r)$$

and G is flat, we are reduced to showing that the canonical map $\mathcal{O}_S \to \mathcal{O}_G^{inv}$ is bijective. If we consider translation by the the identity

$$G \times G \to G \times G, (g, h) \to (g, gh),$$

we see that, if f is a section of $\mathcal{O}_G^{inv}$, then $\mu^*(f) = p_1^*(f)$. Pulling back by $(1_G \times Id_G)^*$, we get $f = p_G^* 1_G^*(f)$ which is what we want.

4.9 Proposition

If G is smooth and if we let

$$\delta := p_2^* - \mu^* + p_1^* : p_{G*}\mathcal{P}_{Gm}(1) \to p_{G^2*}\mathcal{P}_{G^2m}(1),$$

we have an isomorphism of filtered modules

$$p_{G*}\mathcal{I}_{Gm}(1) \cap \ker\delta = \mathcal{I}_{Gm}^{inv}(1) \cap \ker d$$

and therefore also
$$p_{G*}\Omega^1_{Gm} \cap \ker\delta = \Omega^{1inv}_{Gm} \cap \ker d.$$

Proof : If $g \in G(S)$ and $i_g : G \to G \times G, h \mapsto (g, h)$, we have $p_2 \circ i_g = \mathrm{Id}_G, \mu \circ i_g = T_g$ and $p_1 \circ i_g = g \circ p_G$. It follows that
$$i_g^* \circ \delta = (\mathrm{Id} - T_g^*) + p_G^* \circ g^*.$$
If $\xi \in p_{G*}\mathcal{I}_{Gm}(1)$ then, trivially, $g^*(\xi) = 0$ and it follows that $i_g^*(\delta(\xi)) = \xi - T_g^*(\xi)$. Thus, any $\xi \in p_{G*}\mathcal{I}_{Gm}(1) \cap \ker\delta$ is g-invariant. This is true for any g and after any base change. It follows that
$$p_{G*}\mathcal{I}_{Gm}(1) \cap \ker\delta \subset \mathcal{I}^{inv}_{Gm}(1).$$
By functoriality, there is a commutative diagram
$$\begin{array}{ccc} \mathcal{P}_{1m}(1) & \xrightarrow{d} & \mathcal{P}_{1m}(2) \\ \downarrow & & \downarrow \\ p_{G*}\mathcal{P}_{Gm}(1) & \xrightarrow{\delta} & p_{G^2*}\mathcal{P}_{G^2m}(1) \end{array}$$
Since we assumed that G is smooth, the vertical arrows induce filtered isomorphisms with the invariant part in the bottom and we obtain
$$\begin{array}{ccc} \mathcal{P}_{1m}(1) & \xrightarrow{d} & \mathcal{P}_{1m}(2) \\ || & & || \\ \mathcal{P}^{inv}_{Gm}(1) & \xrightarrow{\delta} & \mathcal{P}^{inv}_{G^2m}(1) \end{array}$$
from which it follows that $d = \delta$ on $P^{inv}_{Gm}(1)$ and the result follows.

4.10 Notations

The sheaf
$$p_{G*}\Omega^1_{Gm} \cap \ker\delta = \Omega^{1inv}_{Gm} \cap \ker d \simeq \omega^1_{Gm} \cap \ker d \simeq \mathcal{H}^1(\omega^\bullet_{Gm})$$
of closed invariant forms will be generally written $\omega^{(m)}_{G,0}$.

Note that in the case $m = 0$, the situation was a lot simpler since invariant forms are automatically closed, and in particular, the kernel of δ is exactly the sheaf of invariant differential forms.

4.11 Examples

1. If $G = \mathbf{G}_{aS}$ with parameter t, we already know that ω^1_m is the free module on $\bar{t}, \bar{t}^2, \ldots, \bar{t}^{p^m}$ and it is not difficult to verify that $\bar{t} \in \omega^{(m)}_0$. More precisely, the group law is given by $t \mapsto t \otimes 1 + 1 \otimes t$ and it follows that
$$\delta(t^k) = t^k \otimes 1 - (t \otimes 1 + 1 \otimes t)^k + 1 \otimes t^k = \sum_{i=1}^{k-1} \binom{k}{i} t^i \otimes t^{k-i}.$$

But we also see that, if $\operatorname{char} S = p$, then $\bar{t}^{p^j} \in \omega_0^{(m)}$ for all j. Actually, in this case, $\omega_0^{(m)}$ is locally free on the generators $\bar{t}, \bar{t}^p, \ldots \bar{t}^{p^m}$. In particular, we see that the filtration on $\omega_0^{(m)}$ can have length exactly $p^m - 1$.

In the case $S = \operatorname{Spec} \mathbf{Z}/4$ (so that $p = 2$) and $m = 1$, we see that $\delta(\bar{t}^2) = 2\bar{t} \otimes \bar{t} \neq 0$ but that $2\bar{t}^2 \in \omega_0^{(1)}$. Thus, $\omega_0^{(m)}$ is not always locally free.

The opposite map in $\mathbf{G}_a$ is given by $t \mapsto -t$ and the difference is therefore given by $t \mapsto -t \otimes 1 + 1 \otimes t$. It follows that the canonical inclusion $\omega_m^1 \hookrightarrow p_* \Omega_m^1$ sends $\bar{t}$ to dt and therefore, dt is an invariant differential of level m on $\mathbf{G}_{aS}$.

2. If $G = \mathbf{G}_{mS}$ with parameter t, then ω_m^1 is the free module on $\bar{s}, \bar{s}^2, \ldots, \bar{s}^{p^m}$ with $s = t-1$. Let us consider $\log t^{p^m} := \log(1+s)^{p^m} \in \widehat{\mathcal{P}_{1m}(1)}$. Using the fact that the group law is given by $t \mapsto t \otimes t$, it is not difficult to check that $\delta(\log t^{p^m}) = 0$. Actually, this is a purely formal calculation that can be done over $\mathbf{C}$ where this is well known. It follows that

$$\sum_{i=1}^{p^m} \frac{p^m}{i} \bar{s}^i \in \omega_0^{(m)}.$$

Note that, unlike the case $m = 0$, $\operatorname{Fil}^2 \omega_0^{(m)} \neq 0$ in general. For example, if $p = 2$, $m = 1$ and $S = \operatorname{Spec} \mathbf{Z}/4$, then we have $2\bar{s} - \bar{s}^2 \in \operatorname{Fil}^1$ and $2\bar{s}^2 \in \operatorname{Fil}^2$.

Since the inverse on $\mathbf{G}_m$ is given by $t \mapsto t^{-1}$, the difference is given by $t \mapsto t^{-1} \otimes t$. Thus we see that $t - 1$ is sent to $t^{-1} \otimes t - 1 \otimes 1 = t^{-1}(1 \otimes t - t \otimes 1)$ and it follows that the canonical inclusion $\omega_m^1 \hookrightarrow p_* \Omega_m^1$ sends $\bar{s}$ to dt/t. Finally, we get that

$$\sum_{i=1}^{p^m} \frac{p^m}{i} (\frac{dt}{t})^i$$

is an invariant differential of level m on $\mathbf{G}_{mS}$.

3. We assume now $p \neq 2$ and consider the Legendre elliptic curve E given by the equation

$$y^2 = x(x-1)(x-\lambda)$$

over $\mathbf{A}_S^1 \backslash \{0,1\}$. Since we are interested in the behavior at O which is the point at infinity, we make the usual change of coordinates $z = -x/y, w = -1/y$ and the equation becomes

$$w = z(z-w)(z-\lambda w).$$

Thus, $P_{1m}(1)$ is the m-th divided power envelope of $\mathcal{O}_{\mathbf{A}_S^1 \backslash \{0,1\}}[z]$ with respect to z and ω_{Em}^1 is the free module on $\bar{z}, \ldots, \bar{z}^{p^m}$. A quick calculation as explained in Chapter 4 of [11], for example, shows that the group law is given by

$$z \mapsto 1 \otimes z + z \otimes 1 + (\lambda + 1)(z \otimes z^2 + z^2 \otimes z) + \cdots$$

Assume now that $p = 3$ and $m = 1$. Taking into account the symmetry of the above series due to the commutativity of the group law, one sees that higher powers don't play any role and that

$$\delta(\bar{z}) = (\lambda + 1)(\bar{z} \otimes \bar{z}^2 + \bar{z}^2 \otimes \bar{z}),$$

$$\delta(\bar{z}^2) = 2\bar{z} \otimes \bar{z} + (\lambda + 1)(\bar{z}^2 \otimes \bar{z}^2),$$
$$\delta(\bar{z}^3) = -3(\bar{z} \otimes \bar{z}^2 + \bar{z}^2 \otimes \bar{z}).$$

It turns out that $3\bar{z} + (1+\lambda)\bar{z}^3$ is in $\omega^{(1)}_{E,0}$.

Assume moreover that $\operatorname{char} S = 3$. Then we get $(1+\lambda)\bar{z}^3$ which is in Fil^3. It follows from the formulas that $\omega^{(1)}_{E,0}$ is actually free of rank 1 with generator $\bar{z}^3$ outside $\lambda = -1$. However, at the fiber $\lambda = -1$, both $\bar{z}$ and $\bar{z}^3$ are in $\omega^{(1)}_{E,0}$. In other words, the supersingular fiber is characterized by the fact that $\omega^{(1)}_{E,0}$ is of rank 2 in contrast with the general case where it has rank 1. This is a phenomenon that is specific to higher level because in the classical situation, $\omega^{(0)}_{E,0} = \omega_E$ is globally free of rank 1. Note however that in the case $m = 1$ and $p = 3$, then $Fil^{p^m}\omega^{(m)}_{E,0}$ is also globally free of rank 1.

The next question would be to describe the canonical inclusion $\omega^1_m \hookrightarrow p_*\Omega^1_m$. Since taking opposite on E is given by horizontal symmetry, it sends z to $-z$, and it follows that the difference in E is given by

$$z \mapsto 1 \otimes z - z \otimes 1 + (\lambda + 1)(-z \otimes z^2 + z^2 \otimes z) + \cdots$$

When $m = 0$, one can check that $\bar{z}$ is sent to $dx/2y$ but we have been to lazy to do the computations in other cases.

4.12 Proposition

If A is an abelian scheme over S, there is a Hodge spectral sequence

$$E_2^{ij} = R^j p_{A*}\mathcal{O}_A \otimes_{\mathcal{O}_S} \mathcal{H}^i(\omega^\bullet_{Am}) \Rightarrow \mathcal{H}^n_{dRm}(A/S)$$

Proof : Consider the Hodge to de Rham spectral sequence of level m :

$$E_1^{ij} = R^j p_{A*}\Omega^i_{Am} \Rightarrow \mathcal{H}^n_{dRm}(A/S)$$

We know that, for all i, $\Omega^i_{Am} = p_A^*\omega^i_{Am}$. Since we also have $p_{A*}\mathcal{O}_A = \mathcal{O}_S$, we see that

$$R^j p_{A*}\Omega^\bullet_{Am} = R^j p_{A*}\mathcal{O}_A \otimes_{\mathcal{O}_S} \omega^\bullet_{Am}$$

and we get the spectral sequence

$$E_2^{ij} = R^j p_{A*}\mathcal{O}_A \otimes_{\mathcal{O}_S} \mathcal{H}(\omega^\bullet_{Am}) \Rightarrow \mathcal{H}^n_{dRm}(A/S).$$

4.13 Corollary

We have an exact sequence

$$0 \to \omega^{(m)}_{A,0} \to \mathcal{H}^1_{dRm}(A/S) \to \mathcal{R}^1 p_{A*}\mathcal{O}_A$$

Proof : Just note that $\omega^{(m)}_{A,0} = H^1(\omega^\bullet_{Am})$.

4.14 Remark

When S is smooth, using some results from [12], we can show that we actually get a short exact sequence. We hope to be able to prove this in general in a forthcoming article.

5 Crystalline extension groups

In this section, we generalize to higher level the first results of the second chapter of [3].

5.1 Definitions

A category I is said *very small* if $\mathrm{Ob}(I)$ is countable and for each $\alpha, \beta \in \mathrm{Ob}(I)$, $\mathrm{Hom}(\alpha, \beta)$ is finite. A *decoration* on a very small category I is a pair composed by a *degree map* $d : \mathrm{Ob}(I) \to \mathbf{Z}$ with finite fibers and a *sign map* $\epsilon : \mathrm{Arr}(I) \to \{\pm\}$.

5.2 Notations

If Y is any object in a topos $\mathcal{T}$, we will denote by $\mathbf{Z}^{(Y)}$ the free abelian group on Y in $\mathcal{T}$.

Let (I, d, ϵ) be a decorated category and $Y_\bullet := (Y_\alpha, f_\lambda)$ a diagram indexed by I in $\mathcal{T}$. We set

$$C_n(Y_\bullet) := \oplus_{d(\alpha)=n} \mathbf{Z}^{(Y_\alpha)}$$

and let $\delta_n : C_n \to C_{n-1}$ be given by

$$\sum_{\lambda:\alpha\to\beta} \epsilon(\lambda) f_\lambda : \mathbf{Z}^{(Y_\alpha)} \to \mathbf{Z}^{(Y_\beta)}$$

whenever $\deg(\alpha) = n$ and $\deg(\beta) = n - 1$.

5.3 Definition

A diagram $Y_\bullet$ indexed by a decorated category (I, d, ϵ) is *nice* if $C(Y_\bullet)$ is a complex.

5.4 Remark

P. Deligne has shown that any abelian group G in a topos $\mathcal{T}$ has a canonical left resolution $C_\bullet(G)$ that fits in the above setting. More precisely, there exists a decorated category I and a nice diagram $\Delta(G)_\bullet := (G^{n_\alpha}, f_\lambda)$ indexed by I in the subcategory of $\mathcal{T}$ generated by the powers of G, the projections and the group law such that $C_\bullet(G) = C_\bullet(\Delta(G)_\bullet)$.

To the best of our knowledge, this result of Deligne is unpublished, and we can offer no better reference than what is said in [3].

On the other hand, we are mainly interested in the first terms of the complex $C_\bullet(G)$ in which case we may use the description given in section 2.1 of [3] which we now recall for the

reader's convenience. The diagram has only one object G in degree 0. In degree 1, there is also a unique object G^2 and the morphisms $G^2 \to G$ are

$$\begin{array}{ccccc} G^2 & \to & G & \text{sign} \\ (g_1, g_2) & \mapsto & g_1 & - \\ (g_1, g_2) & \mapsto & g_1 + g_2 & + \\ (g_1, g_2) & \mapsto & g_2 & - \end{array}$$

In degree 3, there are two objects, namely G^2 and G^3. The morphisms from degree 3 to degree 2 are given by

$$\begin{array}{cccc} G^2 & \to & G^2 & \text{sign} \\ (g_1, g_2) & \to & (g_1, g_2) & + \\ (g_1, g_2) & \to & (g_2, g_1) & - \end{array}$$

and by

$$\begin{array}{cccc} G^3 & \to & G^2 & \text{sign} \\ (g_1, g_2, g_3) & \mapsto & (g_2, g_3) & - \\ (g_1, g_2, g_3) & \mapsto & (g_1 + g_2, g_3) & + \\ (g_1, g_2, g_3) & \mapsto & (g_1, g_2 + g_3) & - \\ (g_1, g_2, g^3) & \mapsto & (g_1, g_2) & + \end{array}$$

Thus we get

$$C_\bullet(G) = \cdots \to \mathbf{Z}^{(G^3)} \oplus \mathbf{Z}^{(G^2)} \to \mathbf{Z}^{(G^2)} \to \mathbf{Z}^{(G)}$$

and the maps are given by

$$[g_1, g_2] \mapsto -[g_1] + [g_1 + g_2] - [g_2]$$

from degree 1 to degree 0 and

$$\begin{array}{ccl} [g_1, g_2] & \mapsto & [g_1, g_2] - [g_2, g_1] \\ [g_1, g_2, g_3] & \mapsto & -[g_2, g_3] + [g_1 + g_2, g_3] - [g_1, g_2 + g_3] + [g_1, g_2]. \end{array}$$

from degree 2 to degree 1.

5.5 Lemma

Let $Y_\bullet$ be a nice diagram indexed by a decorated category (I, d, ϵ) in some topos $\mathcal{T}$ and E an abelian sheaf in $\mathcal{T}$. Then, there is a spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s j_{Y_\alpha *} j_{Y_\alpha}^{-1} E \Rightarrow \mathcal{E}xt^{r+s}(C(Y_\bullet), E).$$

Proof : If Y is any object of $\mathcal{T}$, we have

$$\mathcal{H}om_{Gr}(\mathbf{Z}^{(Y)}, E) = \mathcal{H}om(Y, E) = j_{Y*} j_Y^{-1} E.$$

It follows that the complex $\mathcal{H}om_{Gr}(C_\bullet(Y_\bullet), E)$ is canonically isomorphic to a complex whose terms are all of the form $\oplus_{\deg \alpha = n} j_{Y_\alpha *} j_{Y_\alpha}^{-1} E$. Since pulling back by localization is exact and preserves injective sheaves, we get the spectral sequence by applying this remark to an injective resolution of E.

Let $(S, \mathfrak{a}, \mathfrak{b})$ be an m-PD-scheme with p locally nilpotent and $p \in \mathfrak{a}$.

Let X be an S-scheme to which the m-PD-structure of S extends.

5.6 Remark

We denote by $\mathrm{Sch}'_{/X}$ the category of X-schemes to which the m-PD-structure of S extends. The Zariski topology on this category is coarser than the canonical topology and we obtain an embedding of $\mathrm{Sch}'_{/X}$ into the corresponding topos $X_{\mathrm{ZAR}'}$. We will now use the big crystalline topos of level m, $(X/S)^{(m)}_{\mathrm{CRIS}}$ that was introduced in [6] (but considering only schemes to which the m-PD structure extends). Composing the embedding $\mathrm{Sch}'_{/X} \hookrightarrow X_{\mathrm{ZAR}'}$ with the canonical map

$$v_{X/S*} : X_{\mathrm{ZAR}'} \to (X/S)^{(m)}_{\mathrm{CRIS}}$$

from section 1.10 of [6] gives a functor

$$\begin{array}{ccc} \mathrm{Sch}'_{/X} & \to & (X/S)^{(m)}_{\mathrm{CRIS}} \\ Y & \mapsto & \underline{Y} \end{array}$$

Note that if $Y \in \mathrm{Sch}'_{/X}$ with structural morphism $f_Y : Y \to X$ then the canonical morphism

$$f_{Y/X} : (Y/S)^{(m)}_{\mathrm{CRIS}} \to (X/S)^{(m)}_{\mathrm{CRIS}}$$

factors as an isomorphism $(Y/S)^{(m)}_{\mathrm{CRIS}} \simeq (X/S)^{(m)}_{\mathrm{CRIS}\,/\underline{Y}}$ followed by the localization map

$$j_{\underline{Y}} : (X/S)^{(m)}_{\mathrm{CRIS}\,/\underline{Y}} \to (X/S)^{(m)}_{\mathrm{CRIS}}.$$

5.7 Proposition

Let $Y_\bullet$ be a nice diagram in $\mathrm{Sch}'_{/X}$ indexed by a decorated category (I, d, ϵ). If E is an abelian sheaf on $\mathrm{CRIS}^{(m)}(X/S)$, there is a spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{Y_\alpha/X*} f^{-1}_{Y_\alpha/X} E \Rightarrow \mathcal{E}xt^{r+s}(C(\underline{Y}_\bullet), E).$$

Proof : Taking into account the previous remark, this immediately follows from lemma 5.5.

5.8 Remarks

Just as in Chapter III, section 4 of [1], if $Y \in \mathrm{Sch}'_X$, there is a morphism of topos

$$(Y/S)^{(m)}_{\mathrm{cris}} \to (X/S)^{(m)}_{\mathrm{CRIS}}$$

whose inverse image functor $E \mapsto E_Y$ might be called restriction. For any sheaf E on $\mathrm{CRIS}^{(m)}(X/S)$ and any morphism $g : Y' \to Y$, there is a canonical transition map $g^{-1}E_Y \to E_{Y'}$ and these data uniquely determine E. An m-crystal on the big site can be defined as an $\mathcal{O}^{(m)}_{X/S}$-module E such that all E_Y are m-crystals and the transition maps induce isomorphisms $g^*E_Y \simeq E_{Y'}$. In particular, the functor $E \mapsto E_X$ is an equivalence of categories between m-crystals on $\mathrm{CRIS}^{(m)}(X/S)$ and m-crystals on $\mathrm{Cris}^{(m)}(X/S)$. Finally, note that any filtered (resp. transversal) $\mathcal{O}^{(m)}_{X/S}$-module $(E, \mathrm{Fil}^\bullet)$ on $\mathrm{CRIS}^{(m)}(X/S)$ restricts for each Y to a filtered (resp. transversal) $\mathcal{O}^{(m)}_{Y/S}$-module $(E_Y, \mathrm{Fil}^\bullet E_Y)$ on $\mathrm{Cris}^{(m)}(Y/S)$.

For future reference, note also that, as for level 0, in which case this is proved in 1.1.16.4 of [3], we have for any abelian sheaf on $\mathrm{CRIS}^{(m)}(Y/S)$,

$$(Rf_{Y/X*}E)_{(U,T)} = Rf_{U\times Y/T*}E_{U\times Y}.$$

5.9 Definition

A *big transversal m-crystal* on X/S is a crystal E on $\mathrm{CRIS}^{(m)}(X/S)$, endowed with a filtration $\mathrm{Fil}^\bullet$ such that for each $Y \in \mathrm{Sch}'_{/X}$, the filtered m-crystal $(E_Y, \mathrm{Fil}^\bullet E_Y)$ is a transversal m-crystal.

5.10 Lemma

Let $Y \in \mathrm{Sch}'_{/X}$ and $(U \hookrightarrow T) \in \mathrm{CRIS}^{(m)}(X/S)$. Let $i : U \times_X Y \hookrightarrow Z$ be an immersion into a smooth Z in $\mathrm{Sch}'_{/T}$. If E is a big transversal m-crystal on X/S, we have a canonical isomorphism

$$(R^s f_{Y/X*} f^{-1}_{Y/X} \mathrm{Fil}^k E)_{(U,T)} = R^s f_{Z*} \mathrm{Fil}^k[(i_{\mathrm{cris}\,*}E_{U\times Y})_Z \otimes \Omega^\bullet_{Z/Tm}].$$

Proof : We already know that

$$(R^s f_{Y/X*} f^{-1}_{Y/X} \mathrm{Fil}^k E)_{(U,T)} = R^s f_{U\times Y/T*} \mathrm{Fil}^k E_{U\times Y}.$$

On the other hand, the filtered Poincaré lemma in level m tells us that

$$\mathrm{Fil}^k i_* Ru_{Y*}E_{U\times Y} \simeq \mathrm{Fil}^k[(i_{\mathrm{cris}\,*}E_{U\times Y})_Z \otimes \Omega^\bullet_{Z/Tm}]$$

Applying $R^s f_{Z*}$ gives

$$R^s f_{U\times Y/T*} \mathrm{Fil}^k E_{U\times Y} \simeq Rf^s_{Z*} \mathrm{Fil}^k[(i_{\mathrm{cris}\,*}E_{U\times Y})_Z \otimes \Omega^\bullet_{Z/Tm}]$$

and we are done.

5.11 Proposition

Let $Y_\bullet$ be a nice diagram in $\mathrm{Sch}'_{/X}$ and $(U \hookrightarrow T) \in \mathrm{CRIS}^{(m)}(X/S)$. Let $Z_\bullet$ be a nice diagram in $\mathrm{Sch}'_{/T}$ with all Z_α smooth and $i_\bullet : U \times_X Y_\bullet \hookrightarrow Z_\bullet$ a compatible family of immersions. If E is a big transversal m-crystal on X/S, we have a canonical spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{Z_\alpha *} \operatorname{Fil}^k[(i_{\alpha\,\mathrm{cris}\,*} E_{U\times Y_\alpha})_{Z_\alpha} \otimes \Omega^\bullet_{Z_\alpha/Tm}] \Rightarrow \mathcal{E}xt^{r+s}(C(\underline{Y}_\bullet), \operatorname{Fil}^k E)_{(U,T)}.$$

Proof : Since taking value on some object (U,T) is an exact functor, we know from proposition 5.7 that there is a spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} [R^s f_{Y_\alpha/X*} f^{-1}_{Y_\alpha/X} \operatorname{Fil}^k E]_{(U,T)} \Rightarrow \mathcal{E}xt^{r+s}(C(\underline{Y}_\bullet), \operatorname{Fil}^k E)_{(U,T)}.$$

It is therefore sufficient to prove that for each α, we have

$$[R^s f_{Y_\alpha/X*} f^{-1}_{Y_\alpha/X} \operatorname{Fil}^k E]_{(U,T)} = R^s f_{Z_\alpha *} \operatorname{Fil}^k[(i_{\alpha\,\mathrm{cris}\,*} E_{U\times Y_\alpha})_{Z_\alpha} \otimes \Omega^\bullet_{Z_\alpha/Tm}].$$

and this follows from the lemma.

Applying this to the nice diagram $\Delta(G)_\bullet$ that, as we remarked in 5.4, gives rise to Deligne's resolution, we immediately get

5.12 Theorem

Let G be a group scheme in $\mathrm{Sch}'_{/X}$ and $(U \hookrightarrow T) \in \mathrm{CRIS}^{(m)}(X/S)$. Let H be a smooth group scheme on T and $i : G_U \hookrightarrow H_U$ an immersion of groups. If E is a big transversal m-crystal on X/S, we have a canonical spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{H^{n_\alpha}*} \operatorname{Fil}^k[(i_{\alpha\,\mathrm{cris}\,*} E_{G_U^{n_\alpha}})_{H^{n_\alpha}} \otimes \Omega^\bullet_{H^{n_\alpha}/Tm}] \Rightarrow \mathcal{E}xt^{r+s}(\underline{G}, \operatorname{Fil}^k E)_{(U,T)}.$$

5.13 Remarks

If E is any m-crystal on X/S, then the trivial effective filtration $\mathcal{I}^{\{k\}}_{X/S}E$ turns it into a big transversal m-crystal and both proposition 5.11 and theorem 5.12 apply. Also, the theorem is still valid if we replace G with a complex of abelian groups.

Note also that, when H is affine, we have, as in theorem 2.1.8 of [3], an isomorphism in the derived category

$$\operatorname{Fil}^k[\bigoplus_{d(\alpha)=\bullet} f_{H^{n_\alpha}*}[(i_{\alpha\,\mathrm{cris}\,*} E_{G_U^{n_\alpha}})_{H^{n_\alpha}} \otimes \Omega^\bullet_{H^{n_\alpha}/Tm}]_s \simeq R\mathcal{H}om(\underline{G}, \operatorname{Fil}^k E)_{(U,T)}.$$

The proof goes exactly as in [3].

5.14 Proposition

Let G be a smooth group scheme in $\mathrm{Sch}'_{/X}$ and $U \in \mathrm{Sch}_{/X}$. If $k > 0$, we have

$$\begin{array}{rcl} \mathcal{H}om(\underline{G}, \mathcal{I}^{\{k\}}_{X/S})_{(U,U)} & = & 0 \\ \mathcal{E}xt^1(\underline{G}, \mathcal{I}^{\{k\}}_{X/S})_{(U,U)} & = & Fil^k \omega^{(m)}_{G_U,0} \end{array}$$

where $\omega^{(m)}_{G_U,0}$ denotes, as in 4.10, the sheaf of closed invariant forms of level m on G_U. In particular,

$$\mathcal{E}xt^1(\underline{G}, \mathcal{I}_{X/S})_{(U,U)} = \omega^{(m)}_{G_U,0}.$$

Proof : We consider the spectral sequence

$$E_1^{r,s} = \bigoplus_{d(\alpha)=r} R^s f_{G_U^{n_\alpha} *} \operatorname{Fil}^k \Omega^\bullet_{G_U^{n_\alpha}/Um} \Rightarrow \mathcal{E}xt^{p+q}(\underline{G}, \mathcal{I}^{\{k\}}_{X/S})_{(U,U)}.$$

Note first that $E_1^{r,s} = 0$ for $r < 0$ or $s < 0$. Actually, since $k > 0$, we have $\operatorname{Fil}^k \mathcal{O}_{G_U^{n_\alpha}} = 0$ and it follows that $E_1^{r,s} = 0$ for $s = 0$ also. Therefore

$$\mathcal{H}om(\underline{G}, \mathcal{I}^{\{k\}}_{X/S})_{(U,U)} = 0$$

and

$$\mathcal{E}xt^1(\underline{G}, \mathcal{I}^{\{k\}}_{X/S})_{(U,U)} = E_2^{0,1} = \ker \delta : E_1^{0,1} \to E_1^{1,1}$$

But we have for all α,

$$R^1 f_{G_U^{n_\alpha} *} \operatorname{Fil}^k \Omega^\bullet_{G_U^{n_\alpha}/Um} = \operatorname{Fil}^k \ker[d : f_{G_U^{n_\alpha} *} \Omega^1_{G_U^{n_\alpha}/Um} \to f_{G_U^{n_\alpha} *} \Omega^2_{G_U^{n_\alpha}/Um}]$$

and it follows that

$$E_2^{0,1} = \operatorname{Fil}^k \ker[d : f_{G_U *} \Omega^1_{G_U/Um} \to f_{G_U *} \Omega^2_{G_U' Um}] \cap \ker[\delta : f_{G_U *} \Omega^1_{G_U/Um} \to f_{G_U^2 *} \Omega^1_{G_U^2/Um}]$$

which is exactly $Fil^k \omega^{(m)}_{G_U,0}$.

5.15 Remark

The proofs we have presented for theorem 5.12 and proposition 5.14 rely on the unpublished theorem of Deligne mentioned in 5.4. We have taken this option because we think it is the most elegant and natural (even more so if one wants to prove the more general form of theorem 5.12 suggested in 5.13). For readers who fill uncomfortable using unpublished results, we should mention that it would not be difficult to avoid Deligne's resolution and work with a partial resolution as in [3], chapter 2, section 1.

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IRMAR, Université de Rennes I, Campus de Beaulieu, F-35042 Rennes Cedex, France
e-mail: bernard.le-stum@univ-rennes1.fr

Departamento de Matemáticas, Universidad Autónoma de Madrid, E-28049 Madrid, Spain
e-mail: adolfo.quiros@uam.es